



THE UNIVERSITY *of* EDINBURGH

Edinburgh Research Explorer

Time-varying generalizations of all-pass filters

Citation for published version:

Bilbao, S 2005, 'Time-varying generalizations of all-pass filters', *IEEE Signal Processing Letters*, vol. 12, no. 5, pp. 376-379.

Link:

[Link to publication record in Edinburgh Research Explorer](#)

Document Version:

Publisher's PDF, also known as Version of record

Published In:

IEEE Signal Processing Letters

Publisher Rights Statement:

© Bilbao, S. (2005). Time-varying generalizations of all-pass filters. *IEEE Signal Processing Letters*, 12(5), 376-379.

General rights

Copyright for the publications made accessible via the Edinburgh Research Explorer is retained by the author(s) and / or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights.

Take down policy

The University of Edinburgh has made every reasonable effort to ensure that Edinburgh Research Explorer content complies with UK legislation. If you believe that the public display of this file breaches copyright please contact openaccess@ed.ac.uk providing details, and we will remove access to the work immediately and investigate your claim.



Time-Varying Generalizations of All-Pass Filters

Stefan Bilbao

Abstract—In many audio applications, digital all-pass filters are of central importance; a key property of such filters is energy (l_2 norm) preservation. In audio effect and sound synthesis algorithms, it is desirable to have filters that behave as all-passes with time-varying characteristics, but direct generalizations of time-invariant designs can lose the important norm-preserving property; for fast parameter variation, large gain increases are possible. We here call attention to some simple time-varying filter structures, based on wave digital filter designs, that do preserve signal energy and that reduce to simple first- and second-order all-pass filters in the time-invariant case.

Index Terms—All-pass filters, digital audio effects, musical sound synthesis, time-varying filters, wave digital filters.

I. INTRODUCTION

ALL-PASS digital filter designs [1] play a fundamental role in almost all areas of audio signal processing. The defining property of such a design is that it possesses unity gain at all frequencies. A corollary is that in the time domain, the squared norm of a sequence is preserved through the filtering operation; in other words, such a filter is energy preserving.

In several important applications, it is necessary to extend the definition of the all-pass filter to the time-varying coefficient case. Certain audio effects [2] rely on this, as do many physics-based musical sound synthesis algorithms (such as digital waveguides as applied to nonlinear strings [3]–[5] and scattering representations of woodwind toneholes [6], [7]). If the time variation of the filter coefficients is slow, then generally, it is safe to treat such a filter as a quasi-static system, though strictly speaking, frequency domain analysis (and the use of terms such as “unity gain,” etc.) is not generally meaningful. For faster time variation (as may be the case for the physical models mentioned above), such analysis can be misleading; large gain variations are a possibility.

Scattering-type filter designs, such as digital waveguides [8], [9] and, in particular, wave digital filters [10], [11], offer a useful set of design tools for time-varying filters; though it of course remains impossible to perform any meaningful frequency-domain analysis, energy-based stability guarantees are still within reach. They have appeared, in particular, in vocal tract modeling in the wave digital filtering context [12], [13] and earlier as lattice filter designs [14]. In this letter, we point out a simple generalization of the first- and second-order all-pass filters to the time-varying coefficient case, suitable for audio applications, using wave digital filters as a starting point. It is worth noting

that these algorithms are quite different in character from algorithms developed by Mourjopolis [15], Zetterberg and Zhang [16], and Välimäki *et al.* [17] for time-varying filters, for which the emphasis is on transient suppression and minimizing distortion.

II. FIRST-ORDER ALL-PASS FILTERS

For a given real input sequence x_n , indexed by integer n , a general first-order all-pass filter is defined by the recursion

$$y_{n+1} = \beta(ax_{n+1} + x_n) - ay_n \quad (1)$$

where y_n is the output sequence, again indexed by integer n . Here, a is a real filter coefficient, assumed constant, and constrained to be of magnitude less than unity for stability. β is a parameter taking on the value 1 or -1 .

The familiar transfer function for this filter, obtained by taking z -transforms, is

$$H(z^{-1}) = \frac{Y(z^{-1})}{X(z^{-1})} = \beta \frac{a + z^{-1}}{1 + az^{-1}} \quad (2)$$

which possesses the well-known property

$$|H(z^{-1})| = 1 \quad \longrightarrow \quad |Y(z^{-1})| = |X(z^{-1})| \quad (3)$$

on the unit circle (i.e., for $z = e^{j\omega}$). Furthermore, through Parseval's relation, we then have that

$$\|y\| = \|x\| \quad (4)$$

where

$$\|f\| = \left(\sum_{n=-\infty}^{\infty} f_n^2 \right)^{1/2} \quad (5)$$

defines the l_2 norm for square-summable sequences f_n (the \cdot refers to summation variable n). In other words, the all-pass filter is norm preserving. [It is, of course, also possible to derive (4) directly from (1) without using frequency domain concepts.]

A. Time-Varying Parameters

The most straightforward approach to extending the all-pass to the time-varying case is simply to make the coefficients a variable, as per [15]. One way of doing this is to use the recursion

$$y_{n+1} = \beta(a_{n+1}x_{n+1} + x_n) - a_{n+1}y_n. \quad (6)$$

A full analysis of this filter is, of course, difficult, but it is easy enough to show that a simple condition such as

$$|a_{n+1}| \leq \epsilon < 1 \quad (7)$$

Manuscript received August 3, 2004; revised November 2, 2004. The associate editor coordinating the review of this manuscript and approving it for publication was Dr. Darren B. Ward.

The author is with the Sonic Arts Research Centre, Queen's University, Belfast, BT7 1NN, U.K. (e-mail: s.bilbao@qub.ac.uk).

Digital Object Identifier 10.1109/LSP.2005.845602

is not sufficient to ensure energy preservation. Consider the case where x_n is simply an impulse, i.e., $x_0 = 1$, and takes on the value zero otherwise. Then, we have

$$\begin{aligned} y_0 &= \beta a_0 \\ y_1 &= \beta(1 - a_0 a_1) \\ &\vdots \\ y_n &= \beta(1 - a_0 a_1) \prod_{j=2}^n (-a_j) \end{aligned}$$

in which case

$$\|y.\|^2 = (a_0)^2 + (1 - a_0 a_1)^2 \left(1 + \sum_{n=2}^{\infty} \prod_{j=2}^n a_j^2 \right).$$

Then, from (7), we have that

$$\|y.\|^2 \leq \epsilon^2 + (1 + \epsilon^2)^2 \sum_{n=0}^{\infty} \epsilon^{2n} = \frac{1 + 3\epsilon^2}{1 - \epsilon^2}.$$

This bound is tight and can be met with equality for filter coefficients $a_n = \epsilon(-1)^n$. For this choice of coefficients, the output energy is greater than the input energy for any choice of $\epsilon < 1$ and becomes unbounded as ϵ approaches 1.

Other obvious attempts to generalize the all-pass also fail; for instance, instead of (6), we could use

$$y_{n+1} = \beta(a_{n+1}x_{n+1} + x_n) - a_n y_n.$$

Such a choice can lead to similar energy growth, under condition (7). [Considering, for example, a unit impulse input and filter coefficients $a_0 = 0$, $a_n = \epsilon(-1)^n$, for $n \geq 1$, we again get unbounded growth of $\|y.\|$ in the limit as ϵ approaches 1.]

B. Wave Digital One-Port

A full introduction to wave digital filtering principles is beyond the scope of this paper—we refer the reader to [11] for an overview and introduction.

Consider the wave digital one-port shown in Fig. 1, which consists of a single two-port adaptor (either series or parallel) terminated on a delay, with or without sign inversion (β takes on the value 1 or -1), corresponding to a wave digital capacitor or inductor, respectively. At the free port, the input is a sequence x_n , and the output is the sequence y_n . At the terminated port, the input to the adaptor is the sequence w_n , and the output is v_n .

The adaptor, in either the series or parallel case, is defined by values at its two ports: M at the free port and M_n at the port connected to the delay element. We assume M to be a constant and M_n to be a time-varying sequence. Both M and M_n are constrained to be strictly positive. We note that we have used the neutral letter M in this case to indicate that these port values may be taken as port resistances (in the case of a series adaptor)

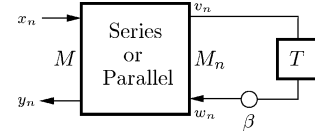


Fig. 1. Wave digital one-port, corresponding to a generalization of a first-order all-pass section.

or as conductances (in the case of a parallel adaptor). Instantaneous scattering at the adaptor can be described by the following matrix equation

$$\begin{bmatrix} y_n \\ v_n \end{bmatrix} = \alpha \begin{bmatrix} \gamma_n & -\sqrt{1 - \gamma_n^2} \\ -\sqrt{1 - \gamma_n^2} & -\gamma_n \end{bmatrix} \begin{bmatrix} x_n \\ w_n \end{bmatrix} \quad (8)$$

where γ_n , the time-varying reflection coefficient, is defined by

$$\gamma_n = \frac{M_n - M}{M_n + M} \quad (9)$$

which, due to the positivity condition on M and M_n , must satisfy

$$|\gamma_n| < 1 \quad (10)$$

for all n . The parameter α is set to 1 for a series junction or -1 for a parallel junction. Note that we have assumed power-normalized scattering here [11]. The reactive one-port is defined simply by

$$w_{n+1} = \beta v_n. \quad (11)$$

The scattering operation, when viewed as a matrix transformation, is easily shown to be orthogonal at each time step n , and thus, we must have

$$y_n^2 + v_n^2 = x_n^2 + w_n^2. \quad (12)$$

It then follows immediately, summing over n , that

$$\|y.\|^2 + \|v.\|^2 = \|x.\|^2 + \|w.\|^2. \quad (13)$$

From the definition (11) of the reactive one-port, we also have that

$$\|v.\|^2 = \|w.\|^2 \quad (14)$$

from which it then follows that

$$\|y.\|^2 = \|x.\|^2. \quad (15)$$

Thus, this wave digital one-port has the same norm-preserving property as the all-pass filter but now in the time-varying case. It is important to note that power normalization is crucial here, in that otherwise (using, say, the more standard voltage wave scattering), the scattering matrix is not orthogonal. The distinction is identical to that between normalized and non-normalized lattice digital filter sections [14].

It is simple enough to arrive at a recursion corresponding to this one-port that relates y_n to x_n directly

$$y_{n+1} = \alpha \gamma_{n+1} x_{n+1} - \beta \sqrt{\frac{1 - \gamma_{n+1}^2}{1 - \gamma_n^2}} (\alpha \gamma_n y_n - x_n). \quad (16)$$

Finally, setting $a_n = \beta \alpha \gamma_n$ gives

$$y_{n+1} = \beta (a_{n+1} x_{n+1} + \phi_n x_n) - \phi_n a_n y_n \quad (17)$$

where

$$\phi_n = \sqrt{\frac{1 - a_{n+1}^2}{1 - a_n^2}} \quad (18)$$

and reduces to (1) when $a_n = a$, a constant.

C. Numerical Example

It is useful to make a comparison of the behavior of the two designs (6) and (17). Consider a simple time variation of the parameter a_n , which takes on the constant value 0.999 for $n \leq 540$ and decreases progressively to a value of -0.999 at $n = 597$; it remains at -0.999 thereafter (see Fig. 2). The input is a sinusoid at 1000 Hz, of amplitude 1; the sample rate is taken to be 44 100 Hz. The output of the filters defined by (6) and (17) are plotted in the middle and bottom panels, respectively, of Fig. 2. Notice that in the case of the time-invariant design, a large transient offset results; it occurs, even though the parameter undergoes a gradual transition over a number of samples. It is important to add that for filter coefficients a_n varying near 1 or -1 , the factor ϕ_n used in the wave digital design (17) can also exhibit strong variation, leading, in this case, to the “kink” observed in the output at the bottom of Fig. 2. Further examination of the effect of the smoothness of the coefficient transition on output smoothness is necessary but cannot be discussed in detail in this letter; neither is there space to discuss the difference in terms of audibility between the two responses, but we do reiterate that stability is the goal here, not necessarily transient suppression.

III. SECOND-ORDER ALLPASS FILTERS

A second-order all-pass filter is defined by

$$y_{n+1} + a y_n + b y_{n-1} = \alpha (b x_{n+1} + a x_n + x_{n-1}). \quad (19)$$

If the coefficients a and b satisfy

$$|a| - 1 < b < 1 \quad (20)$$

then the filter is stable.

Keeping in mind the discussion in the previous section, it should be clear that an appropriate generalized structure for the second-order all-pass will have the form of a wave digital one-port, as shown in Fig. 3. Here, we again read the input and output sequences x_n and y_n from the free port of a three-port adaptor (again, either series or parallel). The other two ports of the adaptor are terminated on a wave digital inductor and a wave digital capacitor. Again, the sequences $M_{1,n}$ and $M_{2,n}$, always

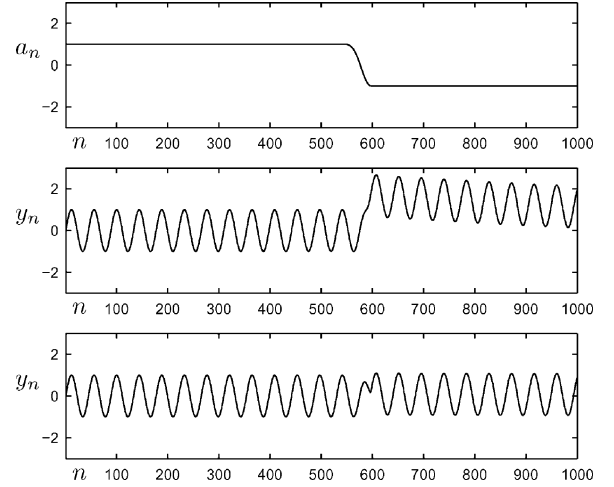


Fig. 2. (Top) Parameter a_n , in a first-order design, plotted against n , the time index. (Middle) Output y_n of filter defined by (6) and (Bottom) of filter defined by (17).

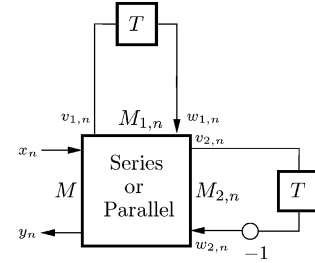


Fig. 3. Wave digital one-port, corresponding to a generalization of a second-order all-pass section.

strictly positive, are to be interpreted as port resistances (for a series adaptor) and as port conductances (for a parallel adaptor). The input and output waves at the two ports are as indicated in the figure. In this case, the scattering equations can be written as

$$\begin{bmatrix} y_n \\ v_{1,n} \\ w_{2,n} \end{bmatrix} = \alpha (\mathbf{I} - \mathbf{q}_n \mathbf{q}_n^T) \begin{bmatrix} x_n \\ w_{1,n} \\ w_{2,n} \end{bmatrix} \quad (21)$$

where $\alpha = 1$ for a series connection, and $\alpha = -1$ for a parallel connection and where the vector \mathbf{q}_n is defined by

$$\mathbf{q}_n = \frac{[\sqrt{2M}, \sqrt{2M_{1,n}}, \sqrt{2M_{2,n}}]^T}{\sqrt{M + M_{1,n} + M_{2,n}}}.$$

In addition, we have

$$w_{1,n} = v_{1,n-1}, \quad w_{2,n} = -v_{2,n-1}. \quad (22)$$

From orthogonality of the scattering operation (21) and using (22), it is again possible to show that the one-port is norm preserving, i.e.,

$$\|y_n\| = \|x_n\|.$$

It is also possible, through tedious algebraic manipulation, to derive a two-step recursion involving only x_n and y_n ; this form is rather complex. In terms of the scattering parameters

$$\gamma_{1,n} = \sqrt{\frac{M_{1,n}}{M}}, \quad \gamma_{2,n} = \sqrt{\frac{M_{2,n}}{M}} \quad (23)$$

and defining the auxiliary parameters

$$\begin{aligned} d_{1,n} &= \frac{(1 + \gamma_{1,n}^2 + \gamma_{2,n}^2)\gamma_{1,n-1}}{2\alpha(\gamma_{2,n}\gamma_{1,n-1} + \gamma_{2,n-1}\gamma_{1,n})} \\ d_{2,n} &= \frac{(-1 + \gamma_{1,n-1}^2 + \gamma_{2,n-1}^2)\gamma_{1,n}}{2(\gamma_{2,n}\gamma_{1,n-1} + \gamma_{2,n-1}\gamma_{1,n})} \\ c_{1,n} &= \frac{(1 - \gamma_{1,n}^2 - \gamma_{2,n}^2)\gamma_{1,n-1}}{2(\gamma_{2,n}\gamma_{1,n-1} + \gamma_{2,n-1}\gamma_{1,n})} \\ c_{2,n} &= \frac{(-1 - \gamma_{1,n-1}^2 - \gamma_{2,n-1}^2)\gamma_{1,n}}{2\alpha(\gamma_{2,n}\gamma_{1,n-1} + \gamma_{2,n-1}\gamma_{1,n})} \end{aligned}$$

the recursion may be written as

$$\begin{aligned} y_{n+1} &+ \frac{d_{2,n+1} + \alpha d_{1,n} - \gamma_{2,n}}{d_{1,n+1}} y_n + \alpha \frac{d_{2,n}}{d_{1,n+1}} y_{n-1} \\ &= -\frac{c_{1,n+1}}{d_{1,n+1}} x_{n+1} - \frac{c_{2,n+1} + \alpha c_{1,n} + \alpha \gamma_{2,n}}{d_{1,n+1}} x_n \\ &\quad - \alpha \frac{c_{2,n}}{d_{1,n+1}} x_{n-1} \end{aligned}$$

If $M_{1,n}$ and $M_{2,n}$ are constant, it reduces to the form (19) with

$$a = 2\alpha \frac{M_1 - M_2}{M + M_1 + M_2}, \quad b = \frac{-M + M_1 + M_2}{M + M_1 + M_2}.$$

a and b , defined as above in terms of the positive quantities M , M_1 , and M_2 , automatically satisfy (20), as expected.

IV. CONCLUSIONS

We have discussed here some simple generalizations of digital all-pass filters to the time-varying case, in particular first- and second-order sections. While in many situations (i.e., when time variation of parameters is slow), simple designs such as (6) and (19) may be used directly, we have shown here a means of extending these designs such that the important energy-preserving property is retained. (We make no claims, however, about the effects on the output in terms of transients and other audible distortion.) Though these designs are wave digital in origin, they may be implemented as recursions just as standard filters are—though, for simplicity, a wave digital realization may be desirable and does not lead to a significant increase in memory or computational requirements. These

designs would appear to be of general applicability throughout all areas of audio signal processing.

ACKNOWLEDGMENT

The author would like to thank J. Pakarinen, M. Karjalainen, C. Erkut, and V. Välimäki of the Acoustics Laboratory at the Helsinki University of Technology for many useful discussions on this topic.

REFERENCES

- [1] P. Regalia, S. Mitra, and P. Vaidyanathan, "The digital all-pass filter: A versatile signal-processing building block," *Proc. IEEE*, vol. 76, no. 1, pp. 19–37, Jan. 1988.
- [2] J. Dattorro, "Effect design part 2: Delay line modulation and chorus," *J. Audio Eng. Soc.*, vol. 45, no. 10, pp. 764–788, 1997.
- [3] V. Välimäki, T. Tolonen, and M. Karjalainen, "Plucked-string synthesis algorithms with tension modulation nonlinearity," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process.*, vol. 2, Phoenix, AZ, Mar. 1999, pp. 977–980.
- [4] C. Erkut, "Aspects in analysis and model-based sound synthesis of plucked string instruments," Ph.D. dissertation, Helsinki Univ. Technol., Lab. Acoust. Audio Signal Process., Finland, 2002.
- [5] T. Tolonen, V. Välimäki, and M. Karjalainen, "Modeling of tension modulation nonlinearity in plucked strings," *IEEE Trans. Speech Audio Process.*, vol. 8, no. 3, pp. 300–310, May 2000.
- [6] G. P. Scavone, "An acoustic analysis of single-reed woodwind instruments with an emphasis on design and performance issues and digital waveguide techniques," Ph.D. dissertation, Department of Music, Stanford Univ., Stanford, CA, 1997.
- [7] M. van Walstijn and G. Scavone, "The wave digital tonehole model," in *Proc. Int. Comput. Music Conf.*, Berlin, Germany, Aug. 2000, pp. 465–468.
- [8] J. O. Smith, III, "Music applications of digital waveguides," Center for Computer Research in Music and Acoustics (CCRMA), Department of Music, Stanford University, Stanford, CA, Tech. Rep. STAN-M-39, 1987.
- [9] —, *Physical Audio Signal Processing*. Stanford, CA: Stanford Univ., 2004.
- [10] A. Fettweis, "Digital filters related to classical structures," *AEU: Archiv für Elektronik und Übertragungstechnik*, vol. 25, pp. 79–89, Feb. 1971.
- [11] —, "Wave digital filters: Theory and practice," *Proc. IEEE*, vol. 74, no. 2, pp. 270–327, Feb. 1986.
- [12] H. Strube, "Time-varying wave digital filters for modeling analog systems," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. ASSP-30, no. 6, pp. 864–88, Dec. 1982.
- [13] —, "Time-varying wave digital filters and vocal-tract models," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process.*, vol. 2, Paris, France, May 3–5, 1982, pp. 923–6.
- [14] A. H. Gray, Jr. and J. D. Markel, "A normalized digital filter structure," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. ASSP-23, no. 3, pp. 268–277, Jun. 1975.
- [15] J. Mourjopolis, E. Kyriakis-Bitzaros, and C. Goutis, "Theory and real-time implementation of time-varying digital audio filters," *J. Audio Eng. Soc.*, vol. 38, no. 7/8, pp. 523–536, 1990.
- [16] L. Zetterberg and Q. Zhang, "Elimination of transients in adaptive filters with applications to speech coding," *Signal Process.*, vol. 15, no. 4, pp. 419–428, 1988.
- [17] V. Välimäki, T. Tolonen, and T. Laakso, "Suppression of transients in time-varying recursive filters for audio signals," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process.*, vol. 6, Seattle, WA, May 1998, pp. 3569–3572.